THE SECOND MOMENT OF $GL(3) \times GL(2)$ L-FUNCTIONS, INTEGRATED

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ABSTRACT. We consider the family of Rankin-Selberg convolution L-functions of a fixed $SL(3,\mathbb{Z})$ Maass form with the family of Hecke-Maass cusp forms on $SL(2,\mathbb{Z})$. We estimate the second moment of this family of L-functions with a "long" integration in t-aspect. These L-functions are distinguished by their high degree (12) and large conductors (of size T^{12}).

1. Introduction

In this paper we study the second moment of the Rankin-Selberg L-functions $L(\phi \times u_j, \frac{1}{2} + it)$ of a fixed Hecke-Maass form ϕ on $SL(3,\mathbb{Z})$ convolved with the family u_j of Hecke-Maass cusp forms on $SL(2,\mathbb{Z})$ as well as with the twists by n^{it}. This family is "large" as measured in a variety of ways: there are T^3 elements in the family, each having degree 12 and conductor of size T^{12}. For comparison, the classical large sieve can estimate the eighth moment of the family of classical Dirichlet L-functions of modulus $q \le Q$ (having Q^2 elements of degree 8 and conductors of size Q^8). The various $GL(2)$ large sieve type inequalities also generally allow for degree 8 L-functions with similarly-sized conductors, so that one can make a case that this family is significantly larger than others appearing in the literature. For use in applications, it is highly desirable to have control over large families of harmonics, as they produce stronger detectors of arithmetical functions; see the introduction of [DuI].

Another way to motivate interest in this particular family is that the first moment (at the central point $s = 1/2$) was recently used by X. Li [Li1] [Li2] to show subconvexity for a self-dual $GL(3)$ L-function in t-aspect (amongst other things). The self-duality is crucially used to impose nonnegativity of the central values. In order to use moments to study non-self-dual forms, as well as Rankin-Selberg convolutions at points other than $s = 1/2$, it seems necessary to study the second moment. However, this approach has substantial new difficulties. In particular, the second moment of this family at the central point has prohibitively large conductors (of size T^{12} compared to T^2 elements in the family, a sixth power). However, one can enlarge the size of the family without substantially growing the size of the conductors by twisting by n^{it} with t almost as large as the spectral parameter. This procedure then brings the problem into the presumably more feasible range where the conductor is the fourth power of the size of the family. Even so, the conductors of the family are still very large so that estimating this moment requires a substantial amount of cancellation. In fact, the main difficulty is showing simultaneous cancellation in the twists by the Hecke-Maass forms as well as by n^{it}. Many of the methods in the literature used to show cancellation in the Maass form aspect are incompatible with the t-aspect integration.

Theorem 1.1. *We have*

$$\int_{-T^{1-\varepsilon}}^{T^{1-\varepsilon}} \sum_{T < t_j \le 2T} |L(u_j \times \phi, \tfrac{1}{2} + it)|^2 dt \ll T^{3+\varepsilon}. \tag{1.1}$$

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Remarks.

- The conductor of $|L(u_j \times \phi, \frac{1}{2}+it)|^2$ is $\asymp T^{12}$, so that the convexity bound is recovered above using the method of Heath-Brown (Lemma 3 of [H-B]).
- An easy consequence of Theorem 1.1 is that for "almost all" t_j,
$$\int_{-T^{1-\varepsilon}}^{T^{1-\varepsilon}} |L(u_j \times \phi, \tfrac{1}{2} + it)|^2 dt \ll T^{1+\varepsilon}.$$
- The reason t is slightly smaller than T is to avoid the intricate dependence of the conductor on t_j, though for $t = t_j$ see the companion paper [Y].
- The method of proof can also handle the analogous twists in the weight aspect by classical holomorphic modular forms on the full modular group; see Section 8 of [ILS].
- The combined t-integral and spectral sum is reminiscent of Sarnak's work on the fourth moment of Grössencharacter L-functions [S2].
- Diaconu, Garrett, and Goldfeld (personal communication) have generalized the method of Good [Goo] to capture quantities of the form (1.1), but with certain weight functions (depending on t_j) in the integral. It is difficult to asymptotically evaluate these weight functions, so it is unknown what this implies about (1.1).
- A. Venkatesh posed this problem during a problem session at the October 2006 AIM workshop on the subconvexity problem.

A natural way to attack this problem is with a hybrid large sieve inequality of the form

$$\int_{-U}^{U} \sum_{t_j \asymp T} \left| \sum_{n \leq N} a_n \lambda_j(n) n^{it} \right|^2 dt \ll \Delta(N, T, U)(NTU)^{\varepsilon} \sum_{n \leq N} |a_n|^2. \tag{1.2}$$

A simple application of Iwaniec's spectral large sieve [Iw1] shows $\Delta(N, T, U) \ll U(N + T^2)$, and one would like to replace this by $N + UT^2$. However, this appears to be an extremely difficult problem, and in fact in this generality it essentially implies the Ramanujan-Petersson conjecture for Maass forms! (To see this, take U large to pick out the diagonal only on the left hand side and choose a_n to select $n = N$ only, showing $|\lambda_j(N)|^2 \ll T^2(TN)^{\varepsilon} \ll_j N^{\varepsilon}$.) One might even consider a simpler problem where U has restricted size (say $U \leq T$) and N is large with respect to T. Even this seems to be a difficult and highly interesting problem (in my opinion). The state of affairs here for $GL(2)$ harmonics is quite different than for $GL(1)$ (multiplicative characters), where we do have the essentially optimal result of Gallagher [Ga]

$$\int_{-U}^{U} \sum_{q \leq Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \pmod{q}} \left| \sum_{n \leq N} a_n \chi(n) n^{it} \right|^2 dt \ll (N + UQ^2) \sum_{n \leq N} |a_n|^2. \tag{1.3}$$

The difficulty of estimating (1.2) for general coefficients a_n is a barrier in our problem, which requires good estimates when the a_n's are specialized to be coefficients of a $GL(3)$ L-function. However, we have more available tools for these specific choices of coefficients, and in particular the $GL(3)$ Voronoi summation formula plays a key role. X. Li [Li1] [Li2] showed how this summation formula can be very powerful in the study of this family, but an attempt to directly generalize her approach on the first moment meets extreme difficulties with the second moment (see the first few sentences of Section 6 below).

In our companion paper [Y], we considered this family of L-functions at the special point $\frac{1}{2} + it_j$. There are some similarities between the two problems but each requires substantially different ideas. In particular, the analog of one of the key ideas in [Y] (namely, Poisson summation in the variable a modulo b in (6.3) below) is not used here as it turned out to

not be substantially helpful, despite my strenuous efforts. Another major difference between these works is that in [Y] we were able to appeal to some large sieve inequalities due to [Lu] (after [DeI]), which we improved further, while here we could not use anything of the form (1.2). Furthermore, the conductors of the family of T^2 elements in [Y] have size T^6 (the cube of the number of elements in the family as opposed to the fourth power appearing here), which has the effect that the $GL(3)$ Voronoi summation formula is relatively less powerful in this article than in the companion. Indeed, the dual sum after Voronoi summation had essentially no length! (Though it should be stressed that this feature only occurred due to our improvement on the relevant large sieve inequality.) This paper is almost entirely independent of [Y] (at the cost of some repetition), though in two places we refer to it for self-contained proofs of facts that are likely to be unsurprising to an expert.

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2. Notation

See [Gol] for the material and notation on $GL(3)$ Maass forms. Suppose ϕ is a Maass form for $SL(3,\mathbb{Z})$ of type $(\nu_1,\nu_2)\in\mathbb{C}^2$ which is an eigenfunction of all the Hecke operators. The Godement-Jacquet L-function associated to ϕ is

$$L(\phi,s)=\sum_{n=1}^{\infty}\frac{A(1,n)}{n^s}=\prod_p(1-A(1,p)p^{-s}+A(p,1)p^{-2s}-p^{-3s})^{-1}. \tag{2.1}$$

Here $A(m,n)$ are the Fourier coefficients normalized as in [Gol]. In particular, $A(1,1)=1$ and $A(m,n)$ are constant on average (see Remark 12.1.8 of [Gol]). The dual Maass form $\widetilde{\phi}$ is of type (ν_2,ν_1) and has $A(n,m)=\overline{A(m,n)}$ as its (m,n)-th Fourier coefficient, whence $L(\widetilde{\phi},s)=\sum_{n=1}^{\infty}A(n,1)n^{-s}$. Letting

$$\Gamma_{\nu_1,\nu_2}(s)=\pi^{-3s/2}\Gamma\left(\frac{s+1-2\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{s+\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{s-1+\nu_1+2\nu_2}{2}\right), \tag{2.2}$$

the functional equation for $L(\phi,s)$ reads

$$\Gamma_{\nu_1,\nu_2}(s)L(\phi,s)=\Gamma_{\nu_2,\nu_1}(1-s)L(\widetilde{\phi},1-s). \tag{2.3}$$

Let (u_j) be an orthonormal basis of Hecke-Maass cusp forms on $SL(2,\mathbb{Z})$ with corresponding Laplace eigenvalues $\frac14+t_j^2$. Let $\lambda_j(n)$ be the Hecke eigenvalue of the n-th Hecke operator for the form u_j. Since the Hecke operators on $GL(2)$ are self-adjoint, the $\lambda_j(n)$'s are real. Then $L(u_j,s)=\sum_{n=1}^{\infty}\lambda_j(n)n^{-s}$ satisfies a functional equation relating to $L(u_j,1-s)$.

As explained in Chapter 12.2 of [Gol], the Rankin-Selberg convolution of ϕ and u_j is

$$L(u_j\times\phi,s)=\sum_{m,n=1}^{\infty}\frac{\lambda_j(n)A(m,n)}{(m^2n)^s}. \tag{2.4}$$

The completed L-function associated to $L(u_j\times\phi,s)$, for u_j even, takes the form

$$\begin{aligned}\Lambda(u_j\times\phi,s)=\pi^{-3s}&\Gamma\left(\frac{s-it_j-\alpha}{2}\right)\Gamma\left(\frac{s-it_j-\beta}{2}\right)\Gamma\left(\frac{s-it_j-\gamma}{2}\right)\\ &\Gamma\left(\frac{s+it_j-\alpha}{2}\right)\Gamma\left(\frac{s+it_j-\beta}{2}\right)\Gamma\left(\frac{s+it_j-\gamma}{2}\right)L(u_j\times\phi,s),\end{aligned} \tag{2.5}$$

where $\alpha = -\nu_1 - 2\nu_2 + 1$, $\beta = -\nu_1 + \nu_2$, and $\gamma = 2\nu_1 + \nu_2 - 1$ (see Theorem 12.3.6 of [Gol] for the explicit gamma factors). Then this Rankin-Selberg convolution has a holomorphic continuation to $s \in \mathbb{C}$ and satisfies the functional equation

$$\Lambda(u_j \times \phi, s) = \Lambda(u_j \times \widetilde{\phi}, 1-s). \tag{2.6}$$

The case with u_j odd is similar, having slightly different constants α, β, γ.

3. Basic tools

3.1. **Approximate functional equation.** We shall use an approximate functional equation to represent the values of L-functions. Write $\lambda_{u_j \times \phi}(n)$ for the coefficient of n^{-s} in the Dirichlet series (2.4) for $L(u_j \times \phi, s)$. Then Theorem 5.3 of [IK] says

$$L(u_j \times \phi, \tfrac12 + it) = \sum_n \frac{\lambda_{u_j \times \phi}(n)}{n^{\frac12 + it}} V_{\frac12 + it}(n) + \epsilon_t \sum_n \frac{\lambda_{u_j \times \widetilde{\phi}}(n)}{n^{\frac12 - it}} V^*_{\frac12 - it}(n), \tag{3.1}$$

where $V_s(y)$ and $V_s^*(y)$ are certain explicit smooth functions, and ϵ_t is a certain complex number of absolute value 1. The weight function $V_{\frac12+it}(x)$ has rapid decay once $n \gg t^{3+\varepsilon}$, on use of Stirling's approximation. More precisely,

$$V_{\frac12 + it}(y) = \frac{1}{2\pi i} \int_{(3)} y^{-s} \frac{\gamma(\frac12 + it + s)}{\gamma(\frac12 + it)} \frac{G(s)}{s} ds, \tag{3.2}$$

where $\Lambda(u_j \times \phi, s) = \gamma(s) L(u_j \times \phi, s)$ and $G(s)$ is an entire function with rapid decay in the imaginary direction. Here $V^*_{\frac12 - it}$ has a similar form to $V_{\frac12+it}$ but with $\gamma(s)$ replaced by $\gamma^*(s)$, where $\Lambda(u_j \times \widetilde{\phi}, s) = \gamma^*(s) L(u_j \times \widetilde{\phi}, s)$.

3.2. **The large sieve.** The classical large sieve inequality for Farey fractions states

$$\sum_{b \le B} \sum_{\substack{x \pmod b \\ (x,b)=1}} \left| \sum_{N \le m < N+M} a_m e\left(\frac{xm}{b}\right) \right|^2 \le (B^2 + M) \sum_{N \le m < N+M} |a_m|^2. \tag{3.3}$$

For our purposes we require an additional oscillatory integral in the spirit of [Ga], but we could not find the following result in the literature. For a self-contained proof, see [Y].

Lemma 3.1. *Let $f(y)$ be a continuously differentiable function on $[N, N+M]$ such that f' does not vanish. Let $X = \sup_{y \in [N,N+M]} \frac{1}{|f'(y)|}$. Then for any complex numbers b_m,*

$$\int_{-T}^{T} \sum_{b \le B} \sum_{\substack{x \pmod b \\ (x,b)=1}} \left| \sum_{N \le m < N+M} b_m e\left(\frac{xm}{b}\right) e(tf(m)) \right|^2 dt \ll (B^2 T + X) \sum_{N \le m < N+M} |b_m|^2. \tag{3.4}$$

3.3. **Kuznetsov formula.** Our tool for summing over the spectrum is the following.

Lemma 3.2 (Kuznetsov)**.** *Suppose that h is holmorphic in the region $|Im(r)| \le \frac12 + \delta$ and satisfies $h(r) = h(-r)$ and $|h(r)| \ll (1+|r|)^{-2-\delta}$ for some $\delta > 0$. Then*

$$\sum_j \alpha_j \lambda_j(m) \lambda_j(n) h(t_j) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir} |\zeta(1+2ir)|^2} h(r) dr \tag{3.5}$$

$$= \pi^{-2} \delta_{m=n} \int_{-\infty}^{\infty} r \tanh(\pi r) h(r) dr + \sum_{c=1}^{\infty} \frac{S(m,n;c)}{c} \check{h}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where

$$\check{h}(x) = \frac{2i}{\pi} \int_{-\infty}^{\infty} \frac{rh(r)}{\cosh(\pi r)} J_{2ir}(x) dr. \tag{3.6}$$

4. Initial cleaning

Using a method similar to that of Section 5 of [Y], we reduce the estimation of (1.1) to

$$H = \int_{-\infty}^{\infty} g\left(\frac{t}{U}\right) \sum_{t_j} \alpha_j h(t_j) \left| \sum_n \frac{\lambda_{u_j \times \phi}(n)}{n^{\frac{1}{2}+it}} w\left(\frac{n}{P}\right) \right|^2 dt, \tag{4.1}$$

where the new notation is as follows: g is a nonnegative, even, Schwartz function satisfying $g(t) \geq 1$ for $|t| \leq 1$ whose Fourier transform is compactly supported; $U = T^{1-\varepsilon}$; $\alpha_j = \frac{|\rho_j(1)|^2}{\cosh(\pi t_j)}$; h is any nonnegative function ≥ 1 for $T < t \leq 2T$; w is a fixed smooth, compactly supported function on $\mathbb{R}^+$; and $P \ll T^{3+\varepsilon}$. Recall that $t_j^{-\varepsilon} \ll \alpha_j \ll t_j^{\varepsilon}$ due to [HL] and [Iw2]. Because of interest in the subconvexity problem, it is desirable to obtain results where $U = T^{1-\delta}$ for some fixed $\delta > 0$. As such, we shall not specialize $U = T^{1-\varepsilon}$ yet.

Writing $\lambda_{u_j \times \phi}(n)$ in terms of $\lambda_j(n)$ and $A(l,n)$ using (2.4), and using Cauchy's inequality to the sum over l, we get that

$$H \ll \log P \sum_{l \ll \sqrt{P}} l^{-1} H_l, \quad H_l = \int_{-\infty}^{\infty} g\left(\frac{t}{U}\right) \sum_{t_j} \alpha_j h(t_j) \left| \sum_n \frac{\lambda_j(n) A(l,n)}{n^{\frac{1}{2}+it}} w\left(\frac{n}{N}\right) \right|^2 dt. \tag{4.2}$$

where $N = P/l^2$. We will show

$$\sum_{l \ll \sqrt{P}} l^{-1} H_l \ll T^{3+\varepsilon} \sum_{l^2 n \ll T^{3+\varepsilon}} \frac{|A(l,n)|^2}{ln}. \tag{4.3}$$

Using the polynomial growth of the Rankin-Selberg convolution $L(\phi \times \phi, s)$, one can show that the inner sum over l and n in (4.3) is bounded by $\log T$ (see Remark 12.1.8 of [Gol]).

One of the basic techniques used throughout this paper is to apply an asymptotic expansion to a particular quantity and reduce the estimation of the entire quantity to that of the leading-order term, as the lower-order terms have all the essential characteristics of the main term yet are of smaller magnitude.

5. Applying the Kuznetsov formula

Our first step in estimating H_l is to apply Lemma 3.2. It is a somewhat involved task to analyze the integral transform $\check{h}$, but Jutila and Motohashi [JM] have obtained a precise asymptotic expansion of $\check{h}$ for the particular choice

$$h(r) = \frac{r^2 + \frac{1}{4}}{T^2} \left[\exp\left(-\left(\frac{r-T}{\Delta}\right)^2 \right) + \exp\left(-\left(\frac{r+T}{\Delta}\right)^2 \right) \right], \tag{5.1}$$

where we take $\Delta = T^{1-\varepsilon}$. Then by (3.19) of [JM], we obtain an asymptotic expansion for $\check{h}(x)$ with leading term

$$\frac{4}{\pi} \sqrt{\frac{2}{x}} \Delta T \exp\left(-\left(\frac{2\Delta T}{x}\right)^2 \right) \cos\left(x - 2T^2 x^{-1} + \frac{\pi}{4} \right). \tag{5.2}$$

It suffices to only treat the leading term since the lower-order terms have the same form as the leading term but are smaller.

Let $H_{l,\Delta}$ be the same expression as H_l but with the choice of h given by (5.1), so that H_l is the sum of $\ll T^\varepsilon$ sums of the form $H_{l,\Delta}$ with T replaced by $T+\Delta$, $T+2\Delta$, etc. Applying the Kuznetsov formula gives that

$$H_{l,\Delta} + \text{Eisenstein} = D_l + K_l, \tag{5.3}$$

say, where D_l corresponds to the diagonal term and K_l is the sum of Kloosterman sums. The Eisenstein contribution is nonnegative and can be discarded for purposes of estimation of $H_{l,\Delta}$. An easy computation gives

$$D_l \ll U\Delta T \sum_{n \ll N} \frac{|A(l,n)|^2}{n}, \tag{5.4}$$

which is sufficient in view of the goal (4.3). Inserting the asymptotic expansion for $\check{h}(x)$ and letting $K_{l,0}$ be the leading term of K_l gives that

(5.5)
$$K_{l,0} \ll \frac{\Delta T}{\sqrt{N}} \int_{-\infty}^{\infty} g\left(\frac{t}{U}\right) \sum_{m,n} a_m \overline{a_n} \left(\frac{m}{n}\right)^{-it} \sum_{c=1}^{\infty} \frac{S(m,n;c)}{\sqrt{c}} e\left(\frac{-2\sqrt{mn}}{c}\right) e^{\frac{iT^2 c}{2\pi\sqrt{mn}}} w_1\left(\frac{c\Delta T}{\sqrt{mn}}\right),$$

where $a_m = A(l,m) m^{-\frac{1}{2}} w_2\left(\frac{m}{N}\right)$; w_2 is a certain smooth, compactly supported function; and $w_1(x)$ is a smooth function on $\mathbb{R}^+$ with rapid decay as $x \to \infty$. Note that the case of the opposite phase can be reduced to an expression of this form.

6. Diophantine approximation

The extreme oscillation of the term $e\left(\frac{-2\sqrt{mn}}{c}\right)$ is a source of difficulty in exploiting cancellation in the sum over m. For instance, an application of the $GL(3)$ Voronoi formula would lead to a sum where the dual variable has size $\approx T^6$ (for "typical" choices of the parameters), which is a catastrophic loss (though there turns out to be a gain in the simplicity of the arithmetical properties of the new sum). It seems necessary to somehow dampen the oscillations of this exponential. To do so, note that the t-integral forces m and n to be close: essentially $m = n(1+O(U^{-1}))$, so that the identity $-2\sqrt{mn} = -m - n + (\sqrt{m}-\sqrt{n})^2$ implies a close approximation (note that $(\sqrt{m}-\sqrt{n})^2 \ll (m-n)^2/N \ll NU^{-2}$). Although $e\left(\frac{-m-n}{c}\right)$ is just as oscillatory as $e\left(\frac{-2\sqrt{mn}}{c}\right)$ (meaning the arguments of the exponential are of the same order of magnitude), it has the property of being periodic in m and n modulo c so that one can treat this term arithmetically and absorb the "remainder" $e\left(\frac{(\sqrt{m}-\sqrt{n})^2}{c}\right)$ into the weight function, which is much less oscillatory. This is the main new idea in this paper. Another pleasant feature of $e\left(\frac{-m-n}{c}\right)$ is that m and n are naturally separated. However, this "twisting" of the Kloosterman sum by the exponential $e\left(\frac{-m-n}{c}\right)$ has the side effect of creating terms of the form $e\left(\frac{(h-1)m}{c}\right)$ where h is coprime to c, so that it does not always hold that $h-1$ is coprime to c (thus making the use of the large sieve inequalities or Voronoi summation problematic). Naturally one can factor out the greatest common divisor of $h-1$ and c to proceed further; that is the content of the following

Lemma 6.1. *For all integers m, n, and positive integers c, we have*

$$S(m,n;c) e\left(\frac{-m-n}{c}\right) = \sum_{ab=c} \sum_{\substack{x \,(\mathrm{mod}\, b) \\ (x(x+a),b)=1}} e\left(\frac{\overline{x}m - (\overline{x+a})n}{b}\right). \tag{6.1}$$

Remark. This calculation seems to have been first performed by Luo [Lu], but see also [IL] for some curious connections.

Proof. By opening the Kloosterman sum, we get that the left hand side of (6.1) is

$$\sum_{\substack{h \,(\mathrm{mod}\ c)\\(h,c)=1}} e\left(\frac{(h-1)m + (\overline{h}-1)n}{c}\right). \tag{6.2}$$

Write $(h-1,c) = a$, $c = ab$, and change variables $h \equiv 1 + ax \pmod{c}$ where now x runs modulo b and satisfies $(x(1+ax), b) = 1$. Note that $\overline{1+ax} - 1 \equiv -ax(\overline{1+ax}) \pmod{b}$. Replacing x by $\overline{x}$ gives (6.1). $\square$

Inserting (6.1) into (5.5) gives

$$K_{l,0} \ll \frac{\Delta T}{\sqrt{N}} \sum_{a,b} \frac{1}{\sqrt{ab}} \sum_{\substack{x \,(\mathrm{mod}\ b)\\(x(x+a),b)=1}} \sum_{m,n} a_m \overline{a_n} e\left(\frac{\overline{x}m - (\overline{x+a})n}{b}\right) Z(m,n), \tag{6.3}$$

where

$$Z(m,n) = e^{\frac{iT^2 ab}{2\pi\sqrt{mn}}} w_1\left(\frac{ab\Delta T}{\sqrt{mn}}\right) e\left(\frac{(\sqrt{m}-\sqrt{n})^2}{ab}\right) \int_{-\infty}^{\infty} g\left(\frac{t}{U}\right)\left(\frac{m}{n}\right)^{it} dt. \tag{6.4}$$

Let

$$J = \sum_{m,n} a_m \overline{a_n} e\left(\frac{\overline{x}m - (\overline{x+a})n}{b}\right) Z(m,n), \tag{6.5}$$

so that

$$K_{l,0} \ll \frac{\Delta T}{\sqrt{N}} \sum_{ab \ll \frac{N}{\Delta T} T^{\varepsilon}} \frac{1}{\sqrt{ab}} \sum_{\substack{x \,(\mathrm{mod}\ b)\\(x(x+a),b)=1}} |J|. \tag{6.6}$$

7. Separation of variables

Next we want to separate the variables m and n in $Z(m,n)$, for which we shall use oscillatory integral transforms. This will allow us to express J in terms of a bilinear form so that we can apply the powerful technology of the large sieve. We will show in this section that $J = \sum_i J_i$ is the sum of $\ll T^{\varepsilon}$ terms of the same form (plus a negligible error term), with leading-order term, say J_1, satisfying

$$|J_1| \ll T^{\varepsilon} \int_{|u| \ll \sqrt{\frac{N}{ab}}} \left| \sum_m a_m e\left(\frac{\overline{x}m}{b}\right) m^{iv} e\left(\frac{u\sqrt{m}}{U\sqrt{ab}}\right) \right| du, \tag{7.1}$$

for some $v \ll T^{\varepsilon}$. Write $|K_{l,0}| \ll \sum_i |K_{l,i}|$ corresponding to this expansion of J; it suffices to treat the leading-order term, say $K_{l,1}$. Applying Cauchy's inequality and changing variables, we obtain the following

Proposition 7.1. *We have*

$$K_{l,1} \ll \Delta T^{1+\varepsilon} \sum_{ab \le \frac{NT^{\varepsilon}}{\Delta T}} \frac{1}{ab} \int_{u \ll 1} \sum_{\substack{x \,(\mathrm{mod}\ b)\\(x,b)=1}} \left| \sum_m a_m e\left(\frac{\overline{x}m}{b}\right) m^{iv} e\left(\frac{u\sqrt{mN}}{Uab}\right) \right|^2 du. \tag{7.2}$$

We shall use different methods of estimation depending on the sizes of a and b. Let $K_1(A,B)$ denote the same sum as $K_{l,1}$ but with $A < a \leq 2A$ and $B < b \leq 2B$, where $AB \ll \frac{NT^\varepsilon}{\Delta T}$. By summing trivially over a, note that

$$K_1(A,B) \ll T^\varepsilon \frac{\Delta T}{B} \int_{u \ll 1} \sum_{b \asymp B} \sum_{\substack{x \ (\mathrm{mod}\ b) \\ (x,b)=1}} \left| \sum_m a_m e\left(\frac{\overline{x} m}{b}\right) m^{iv} e\left(\frac{u\sqrt{mN}}{UAB}\right)\right|^2 du, \tag{7.3}$$

We can immediately apply Lemma 3.1 to get

$$K_1(A,B) \ll T^\varepsilon \Delta T (B + UA) \sum_{m \leq N} |a_m|^2 \ll T^\varepsilon \left(\frac{T^3}{A} + T^2 UA\right) \sum_{m \leq N} |a_m|^2. \tag{7.4}$$

This estimate is sufficient for (4.3) only for A small ($A \ll U^{-1} T^{1+\varepsilon}$), though it simplifies some later work. Also, notice that we used no properties of the coefficients a_m so far.

Proof of (7.1). We will separate the variables in $e^{\frac{iT^2 ab}{2\pi\sqrt{mn}}}$ first using a short Mellin integral, and then use a more involved argument for the remaining expression.

First attach a smooth, compactly-supported weight function $w_3(\sqrt{mn}/N)$ to $Z(m,n)$ that takes the value 1 for all m and n in the support of the implicit weight function appearing in $a_m \overline{a_n}$. Then by Mellin inversion we obtain

$$e^{\frac{iT^2 ab}{2\pi\sqrt{mn}}} w_1\left(\frac{ab\Delta T}{\sqrt{mn}}\right) w_3\left(\frac{\sqrt{mn}}{N}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{a,b,\Delta,T,N}(iv) \left(\frac{N}{\sqrt{mn}}\right)^{iv} dv, \tag{7.5}$$

where

$$F_{a,b,\Delta,T,N}(iv) = \int_0^\infty w_1\left(\frac{ab\Delta T}{Nx}\right) x^{-1} w_3(x) e^{\frac{iT^2 ab}{2\pi N x}} x^{iv} dx. \tag{7.6}$$

If $|v| \gg T^\varepsilon$ then repeated integration by parts shows $F(iv)$ is negligible (meaning smaller than any negative power of T). For $|v| \ll T^\varepsilon$, a trivial bound shows $F(iv) \ll 1$.

Next we separate variables in

$$Z_1(m,n) := e\left(\frac{(\sqrt{m}-\sqrt{n})^2}{ab}\right) \int_{-\infty}^{\infty} g\left(\frac{t}{U}\right) \left(\frac{m}{n}\right)^{it} dt = e\left(\frac{(\sqrt{m}-\sqrt{n})^2}{ab}\right) U\widehat{g}\left(\tfrac{U}{2\pi} \log \frac{m}{n}\right). \tag{7.7}$$

Recalling that $\widehat{g}$ has compact support, which restricts m and n so that $|m-n| \ll U^{-1}N$, we can then use a Taylor expansion to write

$$\log \frac{m}{n} = \log\left(1 + \frac{\sqrt{m}-\sqrt{n}}{\sqrt{n}}\right) = \frac{\sqrt{m}-\sqrt{n}}{\sqrt{n}} - \frac{1}{2}\left(\frac{\sqrt{m}-\sqrt{n}}{\sqrt{n}}\right)^2 + \dots, \tag{7.8}$$

noting $\left|\frac{\sqrt{m}-\sqrt{n}}{\sqrt{n}}\right| \ll U^{-1}$. Then we get an asymptotic expansion for $\widehat{g}$ in the form

$$\widehat{g}\left(\frac{U}{2\pi} \log \frac{m}{n}\right) = \widehat{g}\left(\frac{U}{2\pi} \frac{\sqrt{m}-\sqrt{n}}{\sqrt{n}}\right) - \frac{1}{2} U^{-1} \left(U \frac{\sqrt{m}-\sqrt{n}}{\sqrt{n}}\right)^2 \widehat{g}'\left(\frac{U}{2\pi} \frac{\sqrt{m}-\sqrt{n}}{\sqrt{n}}\right) + \dots. \tag{7.9}$$

Note that each successive term has the same form as the leading term, but is smaller, so we shall only treat the leading term in what follows.

Thus it suffices to consider

$$Z_2(m,n) = e\left(\frac{(\sqrt{m}-\sqrt{n})^2}{ab}\right) U w_4\left(U \frac{\sqrt{m}-\sqrt{n}}{\sqrt{n}}\right), \tag{7.10}$$

where w_4 is smooth and compactly supported. Let $z = \frac{\sqrt{m}-\sqrt{n}}{\sqrt{ab}}$, and $Z = \frac{1}{U}\sqrt{\frac{n}{ab}}$, so that

$$Z_2(m,n) = Ue(z^2)w_4\left(\frac{z}{Z}\right). \tag{7.11}$$

Suppose $Z \gg T^{\varepsilon}$ (in our application this is always satisfied since $ab \ll (\Delta T)^{-1+\varepsilon}N$, $\Delta = T^{1-\varepsilon}$, and $U \ll T^{1-\varepsilon}$). Then by Fourier inversion,

$$Z_2(m,n) = \int_{-\infty}^{\infty} Y(u)e\left(\frac{u(\sqrt{m}-\sqrt{n})}{\sqrt{ab}}\right)du, \quad \text{where} \quad Y(u) = U\int_{-\infty}^{\infty} e(z^2-uz)w_4\left(\frac{z}{Z}\right)dz. \tag{7.12}$$

Next we apply the Plancherel formula $\int_{-\infty}^{\infty} f(x)\widehat{g}(x)dx = \int_{-\infty}^{\infty}\widehat{f}(y)g(y)dy$, with inspiration here from pp. 431-432 of [S1]. We have

$$Y(u) = \frac{e^{\pi i/4}}{\sqrt{2}}UZ\int_{-\infty}^{\infty} e\left(-\left(\frac{u+y}{2}\right)^2\right)\widehat{w_4}(-yZ)dy. \tag{7.13}$$

Expanding the square, we get

$$Y(u) = \sqrt{2}e^{\pi i/4}UZe\left(\frac{-u^2}{4}\right)\int_{-\infty}^{\infty} e\left(-y^2-uy\right)\widehat{w_4}(-2yZ)dy. \tag{7.14}$$

Next truncate the integral at $T^{-\varepsilon}$ (with negligible error), expand $e(-y^2)$ into a Taylor series taking $O(T^{\varepsilon})$ terms so that the remainder is $O(T^{-2009})$, and then extend the integral back to $\mathbb{R}$. This gives an asymptotic expansion for $Y(u)$, with leading-order term say Y_1 given by

$$Y_1(u) = \sqrt{2}e^{\pi i/4}UZe\left(\frac{-u^2}{4}\right)\int_{-\infty}^{\infty} e(-uy)\widehat{w_4}(-2yZ)dy = U\frac{e^{\pi i/4}}{\sqrt{2}}e\left(\frac{-u^2}{4}\right)w_4\left(\frac{u}{2Z}\right). \tag{7.15}$$

The lower-order terms are of a similar shape, but smaller. Thus we conclude that

$$Z_2(m,n) \sim \frac{e^{\pi i/4}}{\sqrt{2}}\int_{-\infty}^{\infty} e\left(\frac{-u^2}{4U^2}\right)w_4\left(\frac{u}{2UZ}\right)e\left(\frac{u(\sqrt{m}-\sqrt{n})}{U\sqrt{ab}}\right)du. \tag{7.16}$$

Hence we get an asymptotic expansion for J with leading-order term J_1 of the form

$$|J_1| \leq \int_{|v|\leq T^{\varepsilon}}\int_u \left|\sum_{m,n} a_m\overline{a_n}(mn)^{-\frac{iv}{2}}w_4\left(\frac{u}{2\sqrt{\frac{n}{ab}}}\right)e\left(\frac{u(\sqrt{m}-\sqrt{n})}{U\sqrt{ab}}\right)\right|dudv. \tag{7.17}$$

Using Cauchy's inequality and estimating the v-integral by the supremum, we get (7.1). $\square$

8. Voronoi summation

In order to improve on (7.4) we resort to use special properties of the coefficients a_n. Our tool is the $GL(3)$ Voronoi summation formula proved by [MS]. We will state this important formula in a form developed by X. Li [Li1].

Theorem 8.1 (Miller-Schmid)**.** *Let ψ be a smooth function with compact support on the positive reals. Then*

$$\begin{aligned}\sum_n A(l,n)e\left(\frac{n\overline{x}}{b}\right)\psi(n) = {}& \frac{b\pi^{-\frac{5}{2}}}{4i}\sum_{n_1|bl}\sum_{n_2>0}\frac{A(n_2,n_1)}{n_1n_2}S\left(lx,n_2;\frac{bl}{n_1}\right)\Psi_1\left(\frac{n_2n_1^2}{b^3l}\right)\\ &+\frac{b\pi^{-\frac{5}{2}}}{4i}\sum_{n_1|bl}\sum_{n_2>0}\frac{A(n_2,n_1)}{n_1n_2}S\left(lx,-n_2;\frac{bl}{n_1}\right)\Psi_2\left(\frac{n_2n_1^2}{b^3l}\right),\end{aligned} \tag{8.1}$$

for certain integral transforms Ψ_1 *and* Ψ_2*.*

We need an explicit asymptotic expansion of Ψ_1 and Ψ_2, which is provided by Lemma 6.1 of [Li2] (generalizing of Lemma 3 of Ivić [Iv]). Each of Ψ_1 and Ψ_2 is a linear combination of two other functions $\Psi_0(x)$ and $x^{-1}\Psi_{0,0}(x)$, say, where each has similar asymptotic behavior, so it suffices to treat $\Psi_0(x)$.

Lemma 8.2 (Ivić, Li)**.** *Suppose* $\psi(x)$ *is supported on* $[N, 2N]$*. Then there exist constants* $c_{j,\pm}$ *such that*

$$\Psi_0(x) = \sum_{j=1}^{L} \sum_{\pm} c_{j,\pm} x \int_0^\infty \psi(y) e(\pm 3(xy)^{1/3}) \frac{dy}{(xy)^{j/3}} + O\left((xN)^{\frac{-L+2}{3}}\right). \tag{8.2}$$

An easy contour shift argument shows that $\Psi_0(x)$ has rapid decay for $xN \to 0$, and is bounded for $xN \ll 1$ (see the original expression (6.12) of [Li2]).

Write (7.3) in the form

$$K_1(A,B) \ll \frac{\Delta T^{1+\varepsilon}}{B} \int_{u \ll 1} \sum_{b \asymp B} \sum_{\substack{x \pmod b \\ (x,b)=1}} |V(u;x,b)|^2 \, du, \tag{8.3}$$

where

$$V(u;x,b) = N^{-\frac{1}{2}} \sum_m A(l,m) e\left(\frac{xm}{b}\right) m^{iv} e\left(\frac{u\sqrt{mN}}{UAB}\right) w_5\left(\frac{m}{N}\right), \tag{8.4}$$

and where $w_5(x) = x^{-\frac{1}{2}} w_2(x)$. Applying the Voronoi summation formula to $V(u;x,b)$ with $\psi(y) = y^{iv} e\left(\frac{u\sqrt{yN}}{UAB}\right) w_5(\frac{y}{N})$, we obtain $V(u;x,b) = V_1(u;x,b) + V_2(u;x,b)$, say, corresponding to the two terms on the right hand side of (8.1). We accordingly write $K_1(A,B) \ll K_{1,1}(A,B) + K_{1,2}(A,B)$. Changing variables by $x \to -x$ shows that $K_{1,2}(A,B)$ is of a form similar to that of $K_{1,1}(A,B)$, so we shall henceforth only treat $K_{1,1}(A,B)$.

First we claim we may assume $xN \gg T^\varepsilon$. Otherwise, using $b \ll \frac{NT^\varepsilon}{AT^2}$, then

$$n_1^2 n_2 \ll \frac{b^3 l}{N} \ll T^\varepsilon \frac{N^2 l}{A^3 T^6}. \tag{8.5}$$

Recalling that $N \ll T^{3+\varepsilon}/l^2$, we get that $n_1^2 n_2 \ll \frac{T^\varepsilon}{l^3 A^3}$. Thus, if $A \gg T^\varepsilon$, then this condition is never satisfied. We already covered the case $A \ll T^\varepsilon$ with (7.4) (since all the variables n_1, n_2, l are small it would be easy to treat this case directly too).

Since $xN \gg T^\varepsilon$, Lemma 8.2 gives an asymptotic expansion of V_1. As usual, we treat the leading-order term, say $K_{0,0}(A,B)$, which takes the form

$$K_{0,0}(A,B) = \frac{\Delta T^{1+\varepsilon}}{B} \int_{u \ll 1} \sum_{b \asymp B} \sum_{\substack{x \pmod b \\ (x,b)=1}} \left| \sum_{n_1 | bl} \sum_{n_2 > 0} \frac{A(n_2,n_1)}{n_1 n_2} S\left(lx, n_2; \frac{bl}{n_1}\right) \Phi\left(\frac{n_2 n_1^2}{b^3 l}\right) \right|^2 du, \tag{8.6}$$

where $\Phi(x)$ is a function of the form

$$\Phi(x) = \frac{b}{\sqrt{N}} x \int_0^\infty w\left(\frac{y}{N}\right) y^{iv} e\left(-3(xy)^{1/3} + \frac{u\sqrt{yN}}{UAB}\right) \frac{dy}{(xy)^{1/3}}, \tag{8.7}$$

with w some smooth, compactly-supported function on the positive reals, and $v \ll T^{\varepsilon}$, possibly after changing variables $u \to -u$ or $v \to -v$. The change of variables $y \to Ny$ gives

$$\Phi(x) = \frac{b}{\sqrt{N}}(xN)^{2/3}N^{iv}\int_0^\infty w(y)y^{-\frac{1}{3}+iv}e\left(-3(xyN)^{1/3} + \frac{u\sqrt{y}N}{UAB}\right)dy. \tag{8.8}$$

Our plan now is to express $K_{0,0}(A,B)$ into a form where we can apply Lemma 3.1; the u-integral is critical for an extra saving effect, and as such it is important to understand the phase of the integral transform Φ (not just its magnitude).

9. Asymptotic behavior of $\Phi(x)$

Under the assumption $xN \gg T^{\varepsilon}$ (with $v \ll T^{\varepsilon/100}$, say), the first term in the exponential in (8.8) dominates over the phase of y^{iv}. Unless the two terms in the exponential are of the same order of magnitude and of opposite signs (in particular, u must be positive), then an easy integration by parts argument shows that $\Phi(x)$ is negligible (smaller than T^{-2009}). That is, $\Phi(x)$ is small unless

$$x \asymp \frac{u^3N^2}{(UAB)^3}. \tag{9.1}$$

Now suppose (9.1) holds.

We shall treat general integrals of the form

$$I = \int_0^\infty g(y)e(\alpha y^{1/2} - \beta y^{1/3})dy, \tag{9.2}$$

where $\alpha, \beta > 0$, $\alpha \asymp \beta$, and g satisfies

$$g \text{ is smooth of compact support on } \mathbb{R}^+, \text{ satisfying } g^{(j)}(y) \ll T_0^j, \tag{9.3}$$

for some parameter $1 \le T_0 \ll |\alpha|^{1/100}$. The stationary phase method easily gives the main term for I, but a search of the literature did not find an adequate asymptotic expansion. In this section we show that I has an asymptotic expansion with leading term equal to

$$I \sim \frac{6\left(\frac{2\beta}{3\alpha}\right)^5}{(2\beta)^{\frac{1}{2}}}e\left(\frac{-4\beta^3}{27\alpha^2} + \frac{1}{8}\right)g\left(\left(\frac{2\beta}{3\alpha}\right)^6\right), \tag{9.4}$$

and where the lower-order terms have the same phase, but are smaller by powers of α.

Applying (9.4) to Φ, we find that

$$\Phi(x) \sim b\sqrt{x}h\left(\frac{(UAB)^3x}{u^3N^2}\right)e\left(\frac{-4x(UAB)^2}{u^2N}\right)\left(\frac{x^2}{u^6}\right)^{iv}z(N,U,A,B), \tag{9.5}$$

where h is a smooth function of compact support on $\mathbb{R}^+$, with bounded derivatives, and $z(N,U,A,B)$ is some bounded function (not depending on either u or x). Noting $b\sqrt{x} = \sqrt{n_1n_2}\frac{1}{\sqrt{\frac{bl}{n_1}}}$ and inserting this expression into (8.6), we obtain

$$K_{0,0}(A,B) \ll \frac{\Delta T^{1+\varepsilon}}{B}\int_{u\ll 1}\sum_{b\asymp B}\sum_{\substack{x \,(\mathrm{mod}\, b)\\(x,b)=1}} \tag{9.6}$$

$$\left|\sum_{n_1|bl}\sum_{n_2}\frac{A(n_2,n_1)}{\sqrt{n_1n_2}}\frac{S\left(lx,n_2;\frac{bl}{n_1}\right)}{\sqrt{\frac{bl}{n_1}}}h\left(\frac{(UAB)^3n_2n_1^2}{u^3b^3lN^2}\right)e\left(\frac{-4n_2n_1^2(UAB)^2}{u^2b^3lN}\right)(n_1n_2^2)^{2iv}\right|^2du.$$

We will continue with this expression in the following section.

Proof of (9.4). Our goal is to use known properties of the Airy function, using ideas similar to those appearing in Section 7. First apply the change of variables $y = t^6$ to get

$$I = \int_0^\infty h(t) e(\alpha t^3 - \beta t^2) dt, \tag{9.7}$$

where $h(t) = g(t^6)(6t^5)$ satisfies (9.3). We will show

$$I \sim \frac{1}{(2\beta)^{\frac{1}{2}}} e\left(\frac{-4\beta^3}{27\alpha^2} + \frac{1}{8}\right) h\left(\frac{2\beta}{3\alpha}\right), \tag{9.8}$$

which immediately implies (9.4).

Next change variables $t \to t + \beta/(3\alpha)$ to get

$$I = e\left(\frac{-2\beta^3}{27\alpha^2}\right) \int_{-\infty}^{\infty} h\left(t + \frac{\beta}{3\alpha}\right) e\left(\alpha t^3 - \frac{\beta^2}{3\alpha} t\right) dt. \tag{9.9}$$

Now use Plancherel with $h_{\alpha,\beta}(t) = h(t + \frac{\beta}{3\alpha})$ (again, $h_{\alpha,\beta}$ satisfies (9.3) except its support may include negative reals) to get

$$I = e\left(\frac{-2\beta^3}{27\alpha^2}\right) \int_{-\infty}^{\infty} \widehat{h_{\alpha,\beta}}(-t) \int_{-\infty}^{\infty} e\left(\alpha y^3 - (\frac{\beta^2}{3\alpha} + t) y\right) dy dt. \tag{9.10}$$

Note

$$\int_{-\infty}^{\infty} e\left(\alpha y^3 - (\frac{\beta^2}{3\alpha} + t) y\right) dy = \frac{2}{(6\pi\alpha)^{1/3}} \int_0^\infty \cos\left(\frac{1}{3} y^3 - 2\pi \frac{\frac{\beta^2}{3\alpha} + t}{(6\pi\alpha)^{1/3}} y\right) dy, \tag{9.11}$$

which is

$$\frac{2\pi}{(6\pi\alpha)^{1/3}} \mathrm{Ai}\left(-2\pi \frac{\frac{\beta^2}{3\alpha} + t}{(6\pi\alpha)^{1/3}}\right), \tag{9.12}$$

where $\mathrm{Ai}(x)$ is the Airy function. Thus

$$I = \frac{2\pi}{(6\pi\alpha)^{1/3}} e\left(\frac{-2\beta^3}{27\alpha^2}\right) \int_{-\infty}^{\infty} \widehat{h_{\alpha,\beta}}(-t) \mathrm{Ai}\left(-2\pi \frac{\frac{\beta^2}{3\alpha} + t}{(6\pi\alpha)^{1/3}}\right) dt. \tag{9.13}$$

Next change variables by $t \to \frac{\beta^2}{3\alpha} t$ to get

$$I = \frac{2\pi}{(6\pi\alpha)^{1/3}} e\left(\frac{-2\beta^3}{27\alpha^2}\right) \int_{-\infty}^{\infty} \frac{\beta^2}{3\alpha} \widehat{h_{\alpha,\beta}}\left(-\frac{\beta^2}{3\alpha} t\right) \mathrm{Ai}\left(-2\pi \frac{\frac{\beta^2}{3\alpha}(1+t)}{(6\pi\alpha)^{1/3}}\right) dt. \tag{9.14}$$

Next we insert the asymptotic expansion for the Airy function at large negative argument (see (4.07) of [O]), namely

$$\mathrm{Ai}(-x) \sim \frac{1}{\sqrt{\pi} x^{\frac{1}{4}}} \left[\cos\left(\frac{2}{3} x^{3/2} - \frac{\pi}{4}\right) \sum_k \frac{c_{2k}}{x^{3k}} + \sin\left(\frac{2}{3} x^{3/2} - \frac{\pi}{4}\right) \sum_k \frac{c_{2k+1}}{x^{\frac{3}{2}(2k+1)}}\right], \tag{9.15}$$

for certain explicit constants c_k (in particular, $c_0 = 0$). Thus

$$I \sim \frac{2\sqrt{\pi}}{(6\pi\alpha)^{1/3}} e\left(\frac{-2\beta^3}{27\alpha^2}\right) \int_{-\infty}^{\infty} \frac{\beta^2}{3\alpha} \widehat{h_{\alpha,\beta}}\left(-\frac{\beta^2}{3\alpha}t\right) \frac{\cos\left(\frac{2}{3}\left(\frac{2\pi\frac{\beta^2}{3\alpha}(1+t)}{(6\pi\alpha)^{1/3}}\right)^{3/2} - \frac{\pi}{4}\right)}{\left(\frac{2\pi\frac{\beta^2}{3\alpha}(1+t)}{(6\pi\alpha)^{1/3}}\right)^{1/4}} dt, \tag{9.16}$$

where the lower-order terms have a similar shape but are multiplied by powers of $\frac{\alpha^2}{\beta^3}(1+t)^{-3/2} \asymp \alpha^{-1}(1+t)^{-3/2}$. The terms with cos replaced by sin are treated similarly, so we work with cos only. This expression simplifies as

$$I \sim \frac{2^{\frac{1}{2}}}{\beta^{\frac{1}{2}}} e\left(\frac{-2\beta^3}{27\alpha^2}\right) \int_{-\infty}^{\infty} \frac{\beta^2}{3\alpha} \widehat{h_{\alpha,\beta}}\left(-\frac{\beta^2}{3\alpha}t\right) (1+t)^{-\frac{1}{4}} \cos\left(\frac{2}{3}\left(\frac{2\pi\frac{\beta^2}{3\alpha}(1+t)}{(6\pi\alpha)^{1/3}}\right)^{3/2} - \frac{\pi}{4}\right) dt. \tag{9.17}$$

The usual integration by parts argument shows $\widehat{h_{\alpha,\beta}}(x) \ll \left(\frac{T_0}{x}\right)^K$ for any $K \geq 0$. Thus we may truncate the integral at $t \ll \alpha^{-\frac{2}{3}}$ at no cost, and expand $(1+t)^{-\frac{1}{4}}$ into a Taylor series.

Now write $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, and write $I \sim I_+ + I_-$ correspondingly. Thus

$$I_\pm \sim \frac{1}{(2\beta)^{\frac{1}{2}}} e\left(\frac{-2\beta^3}{27\alpha^2}\right) \int_{-\alpha^{-2/3}}^{\alpha^{-2/3}} \frac{\beta^2}{3\alpha} \widehat{h_{\alpha,\beta}}\left(-\frac{\beta^2}{3\alpha}t\right) e\left(\pm\left(\frac{2\beta^3}{27\alpha^2}(1+t)^{\frac{3}{2}} - \frac{1}{8}\right)\right) dt. \tag{9.18}$$

Next we take a Taylor series for $(1+t)^{3/2} = 1 + \frac{3}{2}t + \dots$ in the exponential. The quadratic and higher terms are small (much less than 1) so we take a Taylor series expansion for the exponential of these terms, giving another asymptotic expansion with leading-order term

$$I_\pm \sim \frac{e\left(\frac{\mp 1}{8}\right)}{(2\beta)^{\frac{1}{2}}} e\left(\frac{-2\beta^3(1\mp 1)}{27\alpha^2}\right) \int_{-\alpha^{-2/3}}^{\alpha^{-2/3}} \frac{\beta^2}{3\alpha} \widehat{h_{\alpha,\beta}}\left(-\frac{\beta^2}{3\alpha}t\right) e\left(\pm\left(\frac{\beta^3}{9\alpha^2}t\right)\right) dt. \tag{9.19}$$

Extending the integral back to $\mathbb{R}$ and changing variables back via $t \longrightarrow \frac{3\alpha}{\beta^2}t$ gives

$$I_\pm \sim \frac{e\left(\frac{\mp 1}{8}\right)}{(2\beta)^{\frac{1}{2}}} e\left(\frac{-2\beta^3(1\mp 1)}{27\alpha^2}\right) \int_{-\infty}^{\infty} \widehat{h_{\alpha,\beta}}(-t)\, e\left(\pm\left(\frac{\beta}{3\alpha}t\right)\right) dt. \tag{9.20}$$

Calculating the integral in terms of h, we get

$$I_\pm \sim \frac{e\left(\frac{\mp 1}{8}\right)}{(2\beta)^{\frac{1}{2}}} e\left(\frac{-2\beta^3(1\mp 1)}{27\alpha^2}\right) h_{\alpha,\beta}\left(\mp\frac{\beta}{3\alpha}\right). \tag{9.21}$$

Note $h_{\alpha,\beta}\left(\mp\frac{\beta}{3\alpha}\right) = h\left(\frac{\beta}{3\alpha}(1\mp 1)\right)$. Since h has support on the positive reals, we have $h(0) = 0$, in which case only I_- contributes to I, so then

$$I \sim \frac{1}{(2\beta)^{\frac{1}{2}}} e\left(\frac{-4\beta^3}{27\alpha^2} + \frac{1}{8}\right) h\left(\frac{2\beta}{3\alpha}\right). \tag{9.22}$$

One easily checks that this is the expected main term one obtains from stationary phase. $\square$

10. Cleaning

We have reduced the problem of estimating K_l (originally given by (5.3)) to estimating the rather messy expression $K_{0,0}(A,B)$ given by (9.6), in the sense that any bound on $\sup_{A,B} K_{0,0}(A,B)$, times T^{ε}, is also a bound for K_l (recall that the supremum is over $1 \ll AB \ll T^{-2+\varepsilon}N$). We shall make some preliminary transformations to clean up this expression. The reader interested in the essential details should consider the crucial case $u \asymp 1$, $l = n_1 = 1$, which greatly simplifies the forthcoming calculations.

As a first step, apply Cauchy's inequality to take the sum over $n_1 | bl$ outside the absolute values, and divide up the sum over n_2 into dyadic intervals $n_2 \asymp N_2$, where in view of (9.1),

$$N_2 \ll \frac{lN^2}{n_1^2 (UA)^3}. \tag{10.1}$$

Thus

$$K_{0,0}(A,B) \ll T^{\varepsilon} \frac{\Delta T}{B} \sum_{b \asymp B} \sum_{\substack{x \,(\mathrm{mod}\ b) \\ (x,b)=1}} \sum_{n_1 | bl} K', \tag{10.2}$$

where, for certain complex numbers $a(n_1,n_2)$ with the same absolute value as $A(n_2,n_1)$,

$$K' = \int_{u \ll 1} \left| \sum_{n_2 \asymp N_2} \frac{a(n_1,n_2)}{\sqrt{n_1 n_2}} \frac{S\left(lx, n_2; \frac{bl}{n_1}\right)}{\sqrt{\frac{bl}{n_1}}} h\left(\frac{(UAB)^3 n_2 n_1^2}{u^3 b^3 l N^2}\right) e\left(\frac{4 n_2 n_1^2 (UAB)^2}{u^2 b^3 l N}\right) \right|^2 du. \tag{10.3}$$

Next, locate the remaining variables in dyadic segments, say $l \asymp L$ and $n_1 \asymp N_1$. For such l and n_1, change variables in u via

$$u \to \frac{UAB^{3/2} N_2^{1/3} n_1^{2/3}}{b^{3/2} l^{1/3} N^{2/3}} u^{-1/2}, \tag{10.4}$$

to get

$$K' \ll W' \int_{u \asymp 1} \left| \sum_{n_2 \asymp N_2} \frac{a(n_1,n_2)}{\sqrt{n_1 n_2}} \frac{S\left(lx, n_2; \frac{bl}{n_1}\right)}{\sqrt{\frac{bl}{n_1}}} h\left(\left(\frac{bu}{4B}\right)^{\frac{3}{2}} \frac{n_2}{N_2}\right) e\left(\frac{un_2}{W}\right) \right|^2 du, \tag{10.5}$$

where the restriction to $u \asymp 1$ is redundant to the support of h, and

$$W = \frac{N_2^{2/3} B L^{1/3}}{N_1^{2/3} N^{1/3}}, \qquad W' = \frac{UAN_2^{1/3} N_1^{2/3}}{L^{1/3} N^{2/3}}. \tag{10.6}$$

Next separate the variables u, n_2, and b in $h\left(\left(\frac{bu}{B}\right)^{\frac{3}{2}} \frac{n_2}{N_2}\right)$ by using the Mellin inversion formula; since u is already located to be $\asymp 1$ and $n_2 \asymp N_2$, this can be done at no cost (redefining $a(n_1,n_2)$ as necessary). Then we obtain that $\sum_{l \asymp L} K_l$ is bounded by $\sup \mathcal{K}$, say, where the supremum is over the the various parameters A, B, N_1, etc. satisfying the already-imposed constraints (such as $NL^2 \ll T^{3+\varepsilon}$), and where

$$\mathcal{K} = T^{\varepsilon} \frac{\Delta T}{B} \frac{W'}{L} \sum_{l \asymp L} \sum_{b \asymp B} \sum_{\substack{x \,(\mathrm{mod}\ b) \\ (x,b)=1}} \sum_{\substack{n_1 | bl \\ n_1 \asymp N_1}} \int_{u \asymp 1} \left| \sum_{n_2 \asymp N_2} \frac{a(n_1,n_2)}{\sqrt{n_1 n_2}} \frac{S\left(lx, n_2; \frac{bl}{n_1}\right)}{\sqrt{\frac{bl}{n_1}}} e\left(\frac{un_2}{W}\right) \right|^2 du. \tag{10.7}$$

11. Reduction to the large sieve

We now proceed to estimate $\mathcal{K}$ given by (10.7). Let $d = (b, n_1)$ and change variables $b \to db$, $n_1 \to dn_1$ to get

(11.1)
$$\mathcal{K} \ll T^{\varepsilon} \frac{\Delta T}{B} \frac{W'}{L} \sum_{l \asymp L} \int_{u \asymp 1} \sum_{d \ll B} \sum_{b \asymp B/d} \sum_{\substack{x \,(\mathrm{mod}\, bd) \\ (x,bd)=1}} \sum_{\substack{n_1 | l \\ (n_1,b)=1 \\ dn_1 \asymp N_1}} \left| \sum_{n_2} \frac{a(dn_1, n_2)}{\sqrt{dn_1 n_2}} \frac{S\left(lx, n_2; \frac{bl}{n_1}\right)}{\sqrt{\frac{bl}{n_1}}} e\left(\frac{un_2}{W}\right) \right|^2 du.$$

Write $\frac{l}{n_1} = rs$ where $r | b^{\infty}$ (meaning all primes dividing r also divide b) and $(s, b) = 1$. Next change variables to eliminate l and note that the sum over x only depends modulo b to give

$$\mathcal{K} \ll T^{\varepsilon} \frac{\Delta T}{B} \frac{W'}{L} \int_{u \asymp 1} \sum_{d \ll N_1} d \sum_{b \asymp \frac{B}{d}} \sum_{\substack{n_1 rs \asymp L \\ (n_1 s, b)=1 \\ dn_1 \asymp N_1}} \sum_{r | b^{\infty}} \frac{1}{brs} \mathcal{K}_1, \tag{11.2}$$

where

$$\mathcal{K}_1 = \sum_{\substack{x \,(\mathrm{mod}\, b) \\ (x,b)=1}} \left| \sum_{n_2} \frac{a(dn_1, n_2)}{\sqrt{dn_1 n_2}} S\left(n_1 rsx, n_2; brs\right) e\left(\frac{un_2}{W}\right) \right|^2. \tag{11.3}$$

Next we simplify $\mathcal{K}_1$, by showing

$$\mathcal{K}_1 \leq br^2 s \sideset{}{^*}\sum_{h \,(\mathrm{mod}\, bs)} \left| \sum_{r | n_2} \frac{a(dn_1, n_2)}{\sqrt{dn_1 n_2}} e\left(\frac{h \frac{n_2}{r}}{bs}\right) e\left(\frac{un_2}{W}\right) \right|^2. \tag{11.4}$$

Proof of (11.4). From the multiplicativity relation for Kloosterman sums, we obtain

(11.5)
$$S\left(n_1 rsx, n_2; brs\right) = S(n_1 rs\bar{s}x, n_2 \bar{s}, br) S(n_1 rs\overline{br}x, n_2 \overline{br}; s) = S(n_1 r\bar{s}x, n_2, br) S(0, n_2; s),$$

which becomes $S(rx, n_2; br) S(0, n_2; s)$ after the change of variables $x \to s\overline{n_1}x$ (recall n_1 is coprime to br). Thus

$$\mathcal{K}_1 = \sum_{\substack{x \,(\mathrm{mod}\, b) \\ (x,b)=1}} \left| \sum_{m} b_m S\left(rx, m; br\right) \right|^2, \text{ with } b_m = \frac{a(dn_1, m)}{\sqrt{dn_1 m}} S(0, m; s) e\left(\frac{um}{W}\right). \tag{11.6}$$

Next we compute for arbitrary complex numbers c_m,

$$\sum_{x \,(\mathrm{mod}\, b)} \left| \sum_{m} c_m S(rx, m; br) \right|^2 = b \sum_{m_1, m_2} c_{m_1} \overline{c_{m_2}} \sideset{}{^*}\sum_{\substack{h_1, h_2 \,(\mathrm{mod}\, br) \\ h_1 \equiv h_2 \,(\mathrm{mod}\, b)}} e\left(\frac{h_1 m_1 - h_2 m_2}{br}\right). \tag{11.7}$$

Change variables via $h_i = y + bz_i$, $i = 1, 2$, where y runs modulo b and z_i runs modulo r. Since $r | b^{\infty}$, the condition that $(h_i, br) = 1$ is equivalent to $(y, b) = 1$. The sum over z_i vanishes unless $r | m_i$, in which case the sum is r. Thus (11.7) equals

$$br^2 \sum_{r | m_1, m_2} c_{m_1} \overline{c_{m_2}} \sideset{}{^*}\sum_{y \,(\mathrm{mod}\, b)} e\left(\frac{y(\frac{m_1}{r} - \frac{m_2}{r})}{b}\right) = br^2 \sideset{}{^*}\sum_{y \,(\mathrm{mod}\, b)} \left| \sum_{r | m} c_m e\left(\frac{y \frac{m}{r}}{b}\right) \right|^2, \tag{11.8}$$

and hence

$$\mathcal{K}_1 \leq br^2 \sideset{}{^*}\sum_{y \pmod b} \left| \sum_{r|m} c_m e\left(\frac{y\frac{m}{r}}{b}\right) S(0,m;s) \right|^2, \text{ with } c_m = \frac{a(dn_1,m)}{\sqrt{dn_1 m}} e\left(\frac{um}{W}\right). \tag{11.9}$$

By Cauchy's inequality, we get for any complex coefficients b_m that

$$\left| \sum_m b_m S(0,m;s) \right|^2 \leq c \sideset{}{^*}\sum_{h \pmod s} \left| \sum_m b_m e\left(\frac{hm}{s}\right) \right|^2. \tag{11.10}$$

Note that $S(m,0;s) = S(\frac{m}{r},0;s)$ since $(r,s)=1$. Hence

$$\mathcal{K}_1 \leq br^2 s \sideset{}{^*}\sum_{h \pmod s} \sideset{}{^*}\sum_{y \pmod b} \left| \sum_{r|m} c_m e\left(\frac{y\frac{m}{r}}{b}\right) e\left(\frac{h\frac{m}{r}}{s}\right) \right|^2, \tag{11.11}$$

which gives (11.4) using the Chinese remainder theorem. □

Picking back up the chain of reasoning, we insert (11.4) into (11.2), getting

$$\mathcal{K} \ll T^\varepsilon \frac{\Delta T}{B} \frac{W'}{L} \int_{u \asymp 1} \sum_{d \ll N_1} d \sum_{b \asymp \frac{B}{d}} \sum_{\substack{n_1 rs \asymp L \\ (n_1 s, b)=1 \\ dn_1 \asymp N_1}} \sum_{r|b^\infty} r \sideset{}{^*}\sum_{y \pmod{bs}} \left| \sum_{r|n_2} \frac{a(dn_1,n_2)}{\sqrt{dn_1 n_2}} e\left(\frac{y\frac{n_2}{r}}{bs}\right) e\left(\frac{un_2}{W}\right) \right|^2. \tag{11.12}$$

Next relax the condition that $r|b^\infty$ and for convenience locate the variable $s \asymp S$, where $N_1 rS \asymp dL$. In addition change variables $bs \to c$, getting

$$\mathcal{K} \ll T^\varepsilon \frac{\Delta T}{B} \frac{W'}{L} \sum_{d \ll N_1} d \sum_{dn_1 \asymp N_1} \sum_{r \ll \frac{L}{n_1 S}} r \int_{u \asymp 1} \sum_{c \asymp \frac{BS}{d}} \sideset{}{^*}\sum_{h \pmod c} \left| \sum_{r|n_2} \frac{a(dn_1,n_2)}{\sqrt{dn_1 n_2}} e\left(\frac{y\frac{n_2}{r}}{bs}\right) e\left(\frac{un_2}{W}\right) \right|^2, \tag{11.13}$$

for some choice of parameters. Finally, an application of Lemma 3.1 gives

$$\mathcal{K} \ll T^\varepsilon \frac{\Delta T}{B} \frac{W'}{L} \sum_{d \ll N_1} d \sum_{dn_1 \asymp N_1} \left(\left(\frac{BS}{d}\right)^2 + W \right) \sum_{r \ll \frac{L}{n_1 S}} r \sum_{\substack{n_2 \asymp N_2 \\ n_2 \equiv 0 \pmod r}} \frac{|a(dn_1,n_2)|^2}{dn_1 n_2}, \tag{11.14}$$

which is

$$\ll T^\varepsilon \frac{\Delta T}{B} \frac{W'}{S} \left(\frac{B^2 S^2}{N_1} + WN_1 \right) \sum_{(dn_1)^2 n_2 \ll N_1^2 N_2} \frac{|a(dn_1,n_2)|^2}{dn_1 n_2}. \tag{11.15}$$

Note that $W' \ll 1$ from (10.1), and that $W'W = \frac{UABN_2}{N} \leq \frac{BLN}{(UA)^2 N_1^2}$. Thus the expression (11.15) simplifies as

$$\ll T^\varepsilon \Delta T \left(BS + \frac{LN}{(UA)^2} \right) \sum_{n_1^2 n_2 \ll N_1^2 N_2} \frac{|a(n_1,n_2)|^2}{n_1 n_2}. \tag{11.16}$$

Recalling that $B \ll \frac{NT^\varepsilon}{A\Delta T}$ and $L^2 N \ll T^{3+\varepsilon}$ gives that (11.16) is

$$\ll A^{-1} T^{3+\varepsilon} + \frac{T^2}{(UA)^2} T^{3+\varepsilon}. \tag{11.17}$$

This completes the proof of Theorem 1.1, by combining with (7.4) and setting $U = T^{1-\varepsilon}$. Notice that the case $A = U^{-1}T$ is particularly interesting.

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